7- JAN 1922
MADRAS

THE JOURNAL
OF THE
Indian Mathematical Society

Vol. XIII.] OCTOBER 1921. [No. 5.

SOME PITFALLS IN THE THEORY OF GROUPS.

By G. A. Miller.

On page 563 of volume I of the *Encyclopédie des Sciences Mathématiques* it is noted that authors have frequently corrected the results of their predecessors in giving enumerations of substitution groups. Most of these errors were due to the fact that these authors attempted to give such enumerations before the general theory of abstract groups had been sufficiently developed. There are, however, various steps in the latter theory which the beginner should note with great care, and the nature of some of the difficulties involved therein can perhaps be presented in the most attractive manner by directing attention to errors committed at these points by well-known mathematicians.

One such step relates to the possible automorphisms of a metacyclic group of order $p(p-1)$, p being an odd prime number. It is well-known that the quotient group of this group g with respect to its invariant sub-group H of order p is the cyclic group of order $p-1$. This fact results directly from the property that an odd prime number has primitive roots. In every possible automorphism of g every operator of the cyclic group G/H must correspond to itself since H is a characteristic sub-group of g and the operators of two different co-sets of g with respect to H transform the operators of H into different powers module p.

As an instance of errors due to overlooking similar facts, we note the last two exercises which appear on page 69 of Hilton's *Finite Groups*, 1908. In the former of these exercises, the following relations are given:

$$a^{49} = b^7 = c^3 = 1,\ ab = ba^8,\ bc = cb^2,\ ac = ca.$$

It is evident that the operators a and b thus defined generate a group of order 343 which is a special case of the non-abelian group of order

p^m which contains an operator of order p^{m-1}*. The operator c is supposed to transform this group into itself, to transform the operator a into itself, and to transform an operator which transforms an operator which transforms a into its 8th power into an operator which transforms a into its 15th power. As this is clearly impossible the given conditions do not define a group. For similar reasons the conditions

$$a^{43} = b_7 = c^3 = 1,\ ab = ba^4,\ bc = cb^2,\ ac = ca^6$$

which appear in the second of the two exercises noted above do not define a group, and hence the remarks about the sub-groups are meaningless.

There is, however, a much more interesting instance of errors resulting from a failure to observe the principle to which attention has been directed. In 1893, O. Hölder published an article of more than 100 pages in volume 43 of the *Mathematische Annalen* in which he determined the possible groups of various orders, including those of order pq^2, p and q being prime numbers. W. Burnside redetermined these groups without finding any additional ones, and gave an outline of the arguments involved, in both editions of his *Theory of Groups of Finite Order*, 1897, page 132, and 1911, page 75, respectively. Moreover, various authors used the results of Hölder in their determination of groups whose orders involve more prime factors.

It is therefore of peculiar interest to note here that Hölder overlooked an infinite system of groups of order pq^2, when $p-1$ is divisible by q, as a result of failing to make use of the elementary principle noted above. Some details relating to these groups were given by the present writer in a note published in volume 7 (1921) of the *Proceedings of the National Academy of Sciences*.

A much more elementary principle which is sometimes overlooked relates to the order of the product of two commutative operators s_1, s_2. If the orders of s_1 and s_2 are m and n respectively, and if the highest power of any prime number p which divides m is not equal to the highest power of p which divides n, then the order of the product s_1s_2 is divisible by the highest possible power of p which divides either m or n, but by no higher power of p. On the other hand, if the highest power of p which divides m is also the highest power of p which divides n, then the operators s_1 and s_2 can be so chosen that the order of s_1s_2 is divisible by an arbitrary power of p varying from p^0 to this highest power. The order of s_1s_2 cannot be divisible by a prime number which is prime to both m and n.

---

* Cf. Burnside, *Theory of Groups of Finite Order*, 1897, p. 76.

The fact that the elementary principle which has just been stated is often overlooked is made plausible by noting that in the first edition of his *Theory of Groups*, 1897, page 16, Burnside states incorrectly that the order of $s_1 s_2$ is the least common multiple of m and n. In the second edition, page 15, he states correctly that this order is equal to or is a factor of the least common multiple of m and n. The statement found in the preceding paragraph is evidently more general than this statement by Burnside.

A further elementary principle which has often been overlooked relates to the common definition of a group. For instance, in the first edition of volume 2 of H. Weber's *Lehrbuch der Algebra*, 1896, it is stated on page 54 that the *most important* example of a commutative group is presented by the natural numbers when they are combined by ordinary multiplication. It is well-known that these numbers when thus combined do not form a group at all according to the common definition of the term group, as was also noted by Weber in a foot-note which appears on page 4 of the second edition, 1899, of the work just noted. It may be desirable to mention in this connection that on page 60 of this second edition the incorrect statement noted above is replaced by the following: The most important example of a finite commutative group is presented by the residues of the natural numbers when they are combined by multiplication with respect to an arbitrary modulus.

The reader may be inclined to wonder why Weber did not replace the given incorrect statement by the following: The most important example of a commutative group is presented by the ordinary rational numbers, excluding zero, when they are combined by multiplication. This would have required fewer changes in the spirit and in the words than were actually made. Moreover, while this group is almost as old as mathematics itself, Weber's definition noted above involves a comparatively modern mathematical concept and is evidently incomplete, since the total number of the residues does not usually form a group, as is also noted by Weber himself in the discussion of the given definition. It is singular that in his definitions of the *most important* example of an abelian group and the *most important* example of an abelian group of finite order, Weber was not quite accurate, even if we waive the difficulty of deciding what is the most important group.

The preceding criticism of Weber's use of the term group is justified by the fact that he gave in an earlier part of this work a definition of this term which is not in accord with his present use of it. When a writer uses the term group without first defining it, it is difficult to decide whether he uses the term correctly or incorrectly in view of the wide

range of meanings that have been assigned to it by reliable authorities, as may be seen by comparing the definitions found on page 576 of volume 1 and on page 243 of volume 2 of Tome I of the *Encyclopédie des Sciences Mathematiques.* The use of the term finite group in accord with the former of these definitions is, well established ; A. Loewy, however, directed attention to the fact that the term infinite group should not be defined this way [*Archiv der Mathematik and Physik*, vol. 9 (1905), p. 105.] In fact, he called such a definition incorrect notwithstanding the fact that it appears on page 218 of the first volume of *Encyklopedie der Mathematischen Wissenschaften.*

The difficulties involved in the theories to which attention has been directed above may not be serious ; but the fact that such noted authorities on the subject have made errors ought to justify the use of the term 'pitfalls' in these connections. A fundamental pitfall relating to elementary applications of the notion of group was noted at least as early as 1813 by P. Ruffini.* It relates to the fact that while the substitutions, which transform a rational function formally into itself, always constitute a group, those substitutions which transform a function into one having the same numerical value do not necessarily constitute a group. For instance, if x_1, x_2, x_3, x_4 represent the primitive fifth roots of unity, such that $x_r = x_1^r$ $(r = 1, 2, 3, 4)$, then $x_1^3 x_2 = x_2^3 x_4 = x_3^3 x_1 = x_4^3 x_3$, but the 8 substitutions on four letters which transform one of these functions into itself and into each of the other three of these forms do not constitute a group of order 8, as may readily be verified.

It is, however, possible to construct rational functions of these roots, and of any given roots, such that whenever those functions change their formal values, they also change their numerical values. For instance, the polynomial $(x_1 x_4 + x_2 x_3)$ has evidently the value 2, but its other conjugates have different values. In this case, the 8 substitutions which transform one of these functions into one having the same numerical value constitute a group of order 8.

A noteworthy pitfall in the theory of substitution groups relates to imprimitive groups. All the substitutions of such a group which transform each of the systems of imprimitivity into itself constitute an invariant sub-group, but this sub-group may be the identity, and hence it does not follow that an imprimitive group is necessarily composite, as was incorrectly stated to be the case by E. Netto in his well-known book entitled *Substitutionentheorie*, 1882, page 94. The fact that this book has

---

*Cf. H. Burkhardt, *Abhandlungne tur Geschichet zsr Mathematik*, Heft 6, (1892), p. 155. The distinction between formal and numerical values was noted earlier.

been frequently consulted by the beginner has tended to spread a misconception relating to this point, notwithstanding the fact that the error was promptly corrected in volume 22 of the *Mathematische Annalen*, 1883, page 94.

As the term group is frequently employed in the mathematical literature with its non-technical meaning, it is not surprising that in mathematical bibliographies some articles are listed under Group Theory which have no connection with this theory. This is done, for instance on page 66 of volume 1 of the *Subject Index* of the Royal Society of London Catalogue of Scientific Papers 1800—1900, 1908, where reference is given to papers by Jas. Cockle relating to "vanishing groups". Cockle uses the term "groups" in these papers; in its non-technical sense, for sets of terms and his reasoning on these sets is quite foreign to group theory.

If the substitutions of a group g of degree n are applied to a rational function F of the n letters involved in these substitutions, there results a set of functions F_1, F_2, ... F_λ which are transformed according to a transitive substitution group under G. A necessary and sufficient condition that this transitive substitution group and G are simply isomorphic is that the sub-group of G which transforms F_1 into itself involves no invariant sub-group of G besides the identity. When this condition is not satisfied, this transitive substitution group is evidently simply isomorphic with a quotient group of G, whose order is equal to the order of G divided by the order of the largest invariant sub-group of G which appears in the subgroup which transforms F_1 into itself.

In view of the fact that the *Jahrbuch uber die Fortschritte der Mathematik* is a standard reference book, it may not be out of place to note that on page 41 of the first volume of this journal, E. Netto assumes that the given transitive substitution group is *always* simply isomorphic with G. Even when G transforms the functions F_1, F_2, ..., F_λ, according to a group which is simply isomorphic with G, it does not necessarily transform its own corresponding sub-groups according to such a substitution group. In the special case when F_1 is transformed into itself by only the identity of G, the transitive substitution group in question is the regular group which is simply isomorphic with G.

This list of pitfalls in group theory could easily be extended, but the above may suffice to exhibit the most important of those points in this field which the beginner is likely to come across. The number of pitfalls in group theory is probably not unusually large for such a comparatively new subject, but the preceding instances point to the fact that the number is enough to justify the posting of special danger signs, as the present article purports to do.

MATHEMATICAL POINTS.

By S. V. Ramamurty, I.C.S.

1. *The nature of points.*

Euclid is said to have defined a point as that which is "without parts and without magnitude." This definition refers both to the quality and the quantity of a point. An atom cannot be divided into parts consistently with the retention of the quality of an atom. The life of an ant cannot be divided into parts consistently with the retention of the quality of life. When we say that a point is without parts, we mean it has an organic quality which does not admit of sub-division or dissection into parts. To be without magnitude means that it has no size. The conception of the number zero is a relative conception. Zero is that between which and its parts we can perceive no distinction. Hence what is zero relatively to perception of a certain power ceases to be zero when that perception is increased in power. Thus a point is an organic entity of zero size. Hence it is an absolute quantity which is relative to the observer. A point is thus neither absolute nor relative but is relatively absolute.

2. *The classification and structure of points.*

As the observer's power of perception (conception is but a variety of perception) changes, the zero quantity of the point shifts in value. But all values of the zero do not yield a point but only such as also yield an organic quality to the zero. Hence at the stages of the change in the observer's perception where such perception attains an organic quality, the corresponding zero which also gets the reflexion of that organic quality becomes a point. In pre-Leibnitzian Europe, the fundamental entities of the universe were analysed to be Time, Space and Matter. The monad of Leibnitz, the consciousness of Bergson, the duration of Whitehead—all introduce the same new entity, which to keep in touch with ordinary human experience, I call life.

The objective perception of life means the analysis of the self into life and spirit. Standing at the junction of life and spirit, an atom of matter is a point. Standing at the junction of matter and space, an atom of space is a point. Standing at the junction of space and time, an instant of time is a point. Standing sufficiently remote in spirit from its junction with life, an atom of life is a point. Thus based on the analysis, or what is the same thing, the series of successive syntheses, of the universe as time, space, matter, life and spirit, we get four classes of points, viz., points of time, of space, of matter and of life.

I may mention that Dr. Whitehead's *Concept of Nature* (Cambridge University Press, 1920) when carefully read admits of the existence of these four classes. My time is his time. My space is his ; "timeless space." My matter is his "instantaneous space." My life is his "duration." What Whitehead calls a "moment" is a flat three-directional space, *i.e.*, a plane of three independent directions.

The fourth dimensional geometers of Europe confuse consciousness and time, that is, life and time. This mistake occurs in Minkowshi's building up the fourth dimensional world from space and time. It also occurs in Whitehead's *Synthesis of Nature*, though he recognizes that the passage of the mind is not the same as the passage of Nature but is parallel to it and has to it a relation involving a constant of the scale c^2, where c is the velocity of light. The fact is that the 4th dimensional world of Europe is a bundle of three dimensional worlds and not an organic totality of them. Logic deals with bundles of things. Philosophy deals with organic entities. The definitions which Whitehead introduces frequently into his treatment of the "Concept of Nature" are the strings wherewith he binds the bundle. To deal in the simplest manner with four dimensional space, we have to reduce it first to euclidean space. It is possible to *press* a three-dimensional space into a flat three-directional plane which exists in a four-dimensional euclidean space. So too, it is possible to press a four-dimensional space into a flat four-directional plane which exists in a five-dimensional euclidean space.

If then, we reduce a particle of matter into a flat three-directional area, we get a point of a three-directional plane. Such a point is really what Whitehead calls a "punct." His "level" is such a plane and his "rect" is a straight line in that plane.

So too the "event particle" of the Relativists *is an atom of life.* It is four-dimensional and it can be pressed into a point of a flat four-directional plane.

A point of space ordinarily so called is a point of two dimensional flat area.

An instant of time is a point of a one dimensional flat continuum.

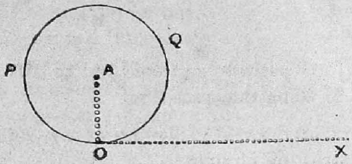
Thus the four classes of points are points of one directional, two directional, three directional and four directional flat spaces. That is to say, the structure of a point depends on the class to which it belongs.

Since the time of Euclid, there has been no organic advance in geometry. The three-dimensional euclidean geometry is a blind lane. Our three-dimensional volumes have no organic quality not inherent in

the plane or the straight line. If geometry were to pass back to the high road of mathematical development, it is necessary to study planes os three independent directions in place of three dimensional spaces—euclidean or non-euclidean.

3. *Applications.*

(1) What happens when a point of matter passes along a *line of space* from one point of space to another such point ?



Let OX be a line of space, *i.e.*, a series of space points. Let POQ be a matter point. A matter point is an atom. Relatively to the scale of size of the human body which is our norm of measurement, an atom is a point. The scale of size of a human body is 10^2 cm. The scale of size of an atom is 10^{-8} cm. Hence we reach the size of the zero, when we pass from 10^2 cm. to 10^{-8} cm. Hence in a rod of unit length, their ratio 10^{-10} represents the size of a point. By 10^{-10}, we mean a number of the scale of size 10^{-10}.

If 1 represents the radius of a matter point, 10^{-10} represents the radius of a space point, 10^{-20} represents the radius of a time-point.

Now c (the maximum) = the velocity of light

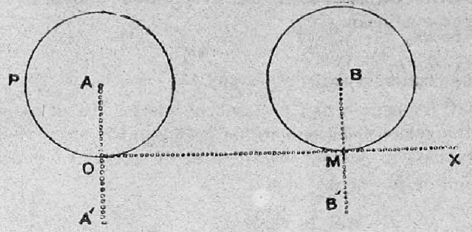
$$= \frac{ds}{dt} = 10^{10} \text{ (i.e., a number of that scale of size).}$$

So then, the matter point POQ (call it P) has in its diameter 10^{10} space points. We say the matter point P is at O, if the space point at O is anywhere in the sphere of radius AO. So that, the number of ways in which a matter point can coincide with a space point is of the order 10^{30}.

Again, let the matter point shift from position O to position M.

Even considering the motion of the centre only and that in a plane, the centre can change from any position on AOA′ to any position on

BOB′, where AO = OA′, BO = OB′ and AOA′ = BOB′ = 10^{10} space points.



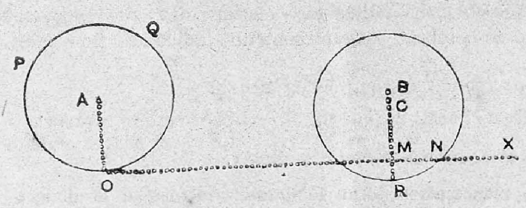
There are $10^{10} \times 10^{10}$ paths to go from AA′ to BB′, *i.e.*, 10^{20} ways of moving from O to M along the space line.

All these rich infinity ways we ignore if we refuse to recognise the classification and structure of points.

(2) I shall now deduce from this relation of matter, space, time points, the well-known relation $m' = m\beta$, for the variation of mass with velocity.

Our measure of matter is that of the height over space. Our measure of space is that of the height over time. Hence the three directions of a piece of three dimensional mass are matter, space and time rods—so measured as heights.

Take first a rod of space.



If OX be the level of time, AO (height above OX) measures space. A space particle at O measures the rod of time perpendicular to AO. If we make this space-time euclidean, the rod becomes the circle POQ and each time point is of the size of a space point.

Let the space rod AO and its circular representation sink into the time mass into the position CR.

Let the circle cut OX in N and CR cut OX in M.

In the first position, AO measures ds in direction perpendicular to OX. In the second position, MB measures ds in direction perpendicular to OX and MN measures ds along of OX.

Let AO be ds_0

ds_0 is the maximum value of ds and $\frac{ds_0}{dt} = c$.

If v be the velocity of the matter residing in the space rod, independently of its velocity in position A, then the direction of v is perpendicular to the direction of ds_0, or AO.

$$\therefore \qquad MN = vdt,$$

and
$$MB = \left(\frac{ds'}{dt}\right) dt,$$

where ds' is the new value of the space rod of the matter, space being measured as height over time.

But
$$MN^2 + MB^2 = BN^2$$
$$= AO^2.$$

$$\therefore \qquad ds' = \sqrt{c^2 - v^2}\ dt.$$

$$\therefore \qquad \frac{ds'}{ds_0} = \sqrt{1 - \frac{v^2}{c^2}} = \beta^{-1}.$$

Take now the particle m moving along a line of space. Its time is unaltered as it moves along a line space. Let m change to m' as ds_0 changes to ds'.

Then
$$m ds_0 = m' ds'$$

$$\therefore \qquad m' = m\beta.$$

(3) Next, I shall deduce the Einsteinian path of a planet.

Let m measure the matter of a planet.

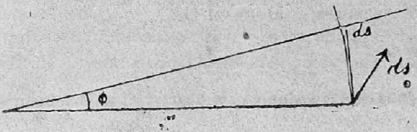
As the planet moves in an euclidean area instead of along a straight line,
$$m ds_0^2 = m' ds'^2.$$

I use now ds as the small change perpendicular to ds_0.

From the equation above
$$m ds_0^2 = m' (ds_0^2 - ds^2).$$

$$\therefore \qquad \frac{m'}{m} = \frac{ds_0^2}{ds_0^2 - ds^2} = \frac{ds_0^2 + ds^2}{ds_0^2},$$

neglecting small quantities of the 4th order.



Hence
$$\frac{m'}{m} = 1 + \frac{ds^2}{ds_0^{\,2}}$$
$$= 1 + \frac{r^2 d\phi^2}{ds_0^{\,2}},$$
$$= 1 + \frac{h^2}{r^2}, \text{ where } h = r^2 \frac{d\phi}{ds_0}.$$

Let M be the matter of the sun. The force between M and m, other things being unaltered, varies as Mm, according to Newton.

If m be considered constant, M changes to M $\left(1 + \frac{h^2}{r}\right)$ in the equation expressing the force. This correction has to be made only in the neighbbourhood of the sun as only then it has an appreciable value. Further this is a change not in space or time, but in matter. The value of the change of matter is expressed in terms of r approximately, but the change is by nature and therefore functionally independent of r. Hence in the integration or differentiation of the equation expressing the attraction between M and m with respect to space or time, the factor $\left(1 + \frac{h^2}{r^4}\right)$ attached to M is unaltered. The Newtonian equations for the path of a planet are

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 = \text{const.} + \left(\frac{2M}{r}\right) \text{ and } r^2\frac{d\phi}{dt} = h,$$

where M is the matter of the sun.

If we make allowance for the change of matter of the planet, which is the same as saying that we make allowance for the change of buoyancy of space, the above equations give (with ds_0 substituted for dt),

$$\left(\frac{dr}{ds_0}\right)^2 + r^2\left(\frac{d\phi}{ds_0}\right)^2 = \text{const.} + \frac{2M}{r}\left(1 + \frac{h^2}{r^2}\right);$$

and
$$r^2\frac{d\phi}{ds_0} = h.$$

These are the Einsteinian equations for the path of a planet.

DECIMALS.

By C. Srinivasa Rao.

[This paper explains some novel methods of reducing vulgar fractions into decimals and also some interesting properties of decimal fractions].

The subject is conveniently divided into three sections, *viz*:

I. To find the number of places in the decimal equivalent of any vulgar fraction;

II. To find the decimal equivalents of fractions with prime denominators (without much division);

III. Some interesting properties of decimals.

Section I.

(1) Every proper fraction is of the form $\frac{a}{x}$, where x is always $> a$, and $\frac{a}{x}$ may be written $a \times \frac{1}{x}$ the value of a ranging from 1 to $x - 1$. To begin with let us take $\frac{1}{x}$.

Here x may be either a prime, or a composite number. The numbers 2 and 5 being the factors of 10, which is the radix for the common scale of notation, every prime number (other than 2 and 5) must end in one of the digits 1, 3, 7 and 9. Every number which ends in any other digits will, of course, be composite. The number of places in the decimal equivalent of $\frac{1}{x}$ depends upon the factors of which x is composed. The figures in the decimal expression for $\frac{1}{x}$ may be either terminating, recurring, or mixed. They will terminate, if x is of the form 2^a, or 5^a, or $(2^a.\ 5^b)$; the number of places will be a in the first two cases and the greater of the two indices in the third case. The reason is obvious and needs no explanation.

(2) Next, let us take up fractions whose decimal equivalents are purely recurring. These are the numbers which end in one of the digits 1, 3, 7 and 9.

(*a*) First let p represent any prime number. The decimal equivalent of $\frac{1}{p}$ is always purely recurring, the number of places being $(p-1)$, or any sub-multiple thereof.

Thus $\frac{1}{3} = \cdot\dot{3}$ contains 1 decimal place recurring, *i.e.* $\frac{3-1}{2}$;

$\frac{1}{7} = \cdot\dot{1}4285\dot{7}$ .... 6 .... $7-1$;

$\frac{1}{11} = \cdot\dot{0}\dot{9}$ .... 2 .... $\frac{11-1}{5}$;

$\frac{1}{13} = \cdot\dot{0}7692\dot{3}$ .... 6 .... $\frac{13-1}{2}$;

$\frac{1}{17} = \cdot\dot{0}58823529411764\dot{7}$ contains 16 places.............. $17-1$;

$\frac{1}{19}, \frac{1}{23}, \frac{1}{29}, \frac{1}{31}, \frac{1}{37}, \frac{1}{41}$, &c., contain, respectively, $19-1$, $23-1$, $29-1$, $\frac{31-1}{2}$, $\frac{37-1}{12}$, $\frac{41-1}{8}$, &c., places.

[There is a sort of a law depending on Fermat's theorem

$$10^{p-1} - 1 \equiv 0\ (p);$$

At times we get

$$10^{\frac{p-1}{t}} - 1 \equiv 0\ (p)].$$

(*b*) Secondly, let us consider fractions of the form $\frac{1}{p^x}$; suppose the decimal equivalent of $\frac{1}{p}$ contains d digits, then $\frac{1}{p^x}$ contains $d \cdot p^{x-1}$ digits except when $p=3$, in which case the number will be $d \cdot 3^{x-2}$.

For example the decimal equivalents of $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \frac{1}{3^5}, \frac{1}{3^6}$, etc. contain respectively 1, 1, 3, 9, 27, 81, etc., places.

$\frac{1}{7}, \frac{1}{7^2}, \frac{1}{7^3}$ contain respectively 6, $6 \times 7^{2-1}$, $6 \times 7^{3-1}$, places;

$\frac{1}{11}, \frac{1}{11^2}, \frac{1}{11^3}$ etc., and $\frac{1}{13}, \frac{1}{13^2}$, etc., contain respectively 2, $2 \times 11^{2-1}$, $2 \times 11^{3-1}$, etc. and 6, $6 \times 13^{2-1}$, &c., places.

(*c*) Thirdly, let us take up fractions of the form $\frac{1}{c}$, where c is a composite number ending in one of the digits 1, 3, 7, and 9. Suppose

$$c = p_1 \times p_2 \times p_3 \times p_4 \ldots\ldots\ldots\ldots$$

all different prime factors. Also suppose the decimal equivalents of $\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \frac{1}{p_4}$ contain d_1, d_2, d_3, d_4 digits, respectively. Then the number of places in the decimal equivalent of $\frac{1}{c}$ will be the L. C. M. of d_1, d_2, d_3, d_4.

For example :—

$\frac{1}{21} = \frac{1}{3 \times 7} = \cdot\dot{0}4761\dot{9}$; 6 being the L.C.M. of the numbers of places in the decimal equivalents of $\frac{1}{3}$ and $\frac{1}{7}$;

$\frac{1}{33} = \frac{1}{3 \times 11} = \cdot\dot{0}\dot{3}$; ...... 2 ...... 1 and 2 ... ... $\frac{1}{3}$ and $\frac{1}{11}$;

$\frac{1}{39} = \frac{1}{3 \times 13} = \cdot\dot{0}2564\dot{1}$; ... 6 ...... 1 and 6 ... ... $\frac{1}{3}$ and $\frac{1}{13}$;

$\frac{1}{119} = \frac{1}{7 \times 17} = \cdot\dot{0}0840336134453781512605042016806722689075630252\dot{1}$,

48 being the L.C.M. of 6 and 16 the numbers corresponding to $\frac{1}{7}$ and $\frac{1}{17}$.

When two or more of the prime factors are the same, we must also take into consideration the rule given under (*b*). For example :—

$\frac{1}{3^4}$ or $\frac{1}{81}$ contains 9 digits and is $\cdot\dot{0}1234567\dot{9}$

$\frac{1}{3^4 \times 11}$ or $\frac{1}{891}$ „ 9×2 (*i.e.* L.C.M. of 9 and 2) digits and is $\cdot\dot{0}0112233445566778\dot{9}$

$\frac{1}{3^5 \times 11}$ or $\frac{1}{2673}$ „ 27×2 (*i.e.* L.C.M. of 27 and 2) digits and is $\cdot\dot{0}0037411148522259633370744481855592966704077815188926\dot{3}$

(*d*) Lastly, let us take fractions of the form $\frac{1}{d}$, where d is a composite number ending in one of the digits 0, 2, 4, 5, 6 and 8. The decimal equivalents of all these fractions are mixed.

In this case $\frac{1}{d}$ takes the form $\frac{1}{2^a 5^b} \times \frac{1}{p}$ or $\frac{1}{2^a 5^b} \times \frac{1}{c}$.

Therefore, in the decimal equivalent of $\frac{1}{d}$ the number of terminating places should be determined by the factor $\frac{1}{2^a\, 5^b}$, and the number of recurring places should be determined by the factor $\frac{1}{p}$, or $\frac{1}{c}$.

Thus, for example :—

$$\frac{1}{18} = \cdot 0\dot{5}\,;\ \frac{1}{30} = .\,0\dot{3}\,;\ \frac{1}{36} = \cdot 02\dot{7}\,;\ \frac{1}{42} = .\,0\dot{2}3809\dot{5}\,;$$

$$\frac{1}{45} = \cdot 0\dot{2}\,;\ \frac{1}{78} = \cdot\,0\dot{1}2820\dot{5},\ \text{etc.}$$

SECTION II.

In this section, we shall see how to write down the decimal equivalent of a given fraction without much labour.

It has been already said above that all the prime numbers must end in one of the digits 1, 3, 7, and 9.

Let us take fractions with prime denominators ending with 9. We shall confine our attention to denominators below 200. They are :—

(*a*) 19, 29, 59, 109, 149 and 179, the corresponding fractions contain $p - 1$ digits.

(*b*) 89 and 199 ………… $\frac{p-1}{3}$ digits.

(*c*) 139 ………… $\frac{p-1}{3}$ digits.

(*d*) 79 ………… $\frac{p-1}{6}$ digits.

(*e*) 9, 49, and 169, being the powers of 3, 7 and 13.

(*f*) 39, 69, 99, 119, 129, 159 and 189 being composite numbers.

(1) Taking any number in the first group, say 19, consider any proper fraction with 19 for the denominator, say $\frac{15}{19}$. Then proceed according to the following rule :—

To the number ending in 9 add 1 and then divide it by 10 and make this result a continued divisor; start with the numerator and divide it by this continued divisor, placing the quotient in the units place and the remainder in the tens place; proceed in this manner till you get the numerator again. The digits in the units place of all the numbers in the order in which they are obtained omitting the starting number will give the decimal equivalent of the given fraction.

Coming to the example in question, *viz.*, $\frac{15}{19}$, if we add 1 to 19 and then divide it by 10, we get 2 which becomes the continued divisor; starting with 15, if we divide it by 2, we get 7 for the quotient to be placed in the units place and 1 for the remainder to be placed in the tens place, thus forming 17 which is the next number after 15; continuing the same process, we get the following numbers : —

15, 17, 18, 9, 14, 7, 13, 16, 8, 4, 2, 1, 10, 5, 12, 6, 3, 11, 15.

$\therefore \quad \frac{15}{19} = .\dot{7}8947368421052631\dot{5}.$

The labour may be further minimised in the following manner :—

Take 15 and then place a decimal point after it ; divide $\frac{15}{2}$ and place the quotient 7 after the decimal point and place the remainder 1 over 5 ; again divide the combination of 1 and 7 *i.e.* 17 by 2 and place the quotient 8 after 7 and the remainder 1 over 7 ; again divide the combination of 1 and 8, *i e.*, 18 by 2 and place the quotient 9 after 8 and the remainder 0 over 8 ; again divide the combination of 0 and 9, *i.e*, 9 by 2 and place the quotient 4 after 9 and the remainder 1 over 9 and so on until you get the combination of 1 and 5, *i.e.*, 15 which is the starting number. The starting number 15 and the upper line of numbers which are the respective remainders may be omitted. Place the recurring signs, *i.e.*, the dots over the first digit 7 and the last digit 5. This whole expression is the decimal equivalent of $\frac{15}{19}$; thus :—

$$\frac{15}{19} = \begin{array}{l} 1\ \ 10101100001010011\cdot \\ 15.\dot{7}89473684210526315\cdot \end{array}$$

To save labour and to avoid mistakes this operation will be adopted in the rest of the work.

To familiarise the reader with the process, two more examples are added :—

(i) $\frac{1}{129}$, (ii) $\frac{17}{189}$, 13 and 19 being the continued divisors.

$$\frac{1}{129} = \begin{array}{l} 1\ \ \ 1096212410121056127135041 0\cdot \\ 1.\ \dot{0}077\,5\,1\,9\,3\,7\,98\,4\,4961\,24031\ ; \end{array}$$

$$\frac{17}{189} = \begin{array}{l} 17\ \ 18178131. \\ 17.\ \dot{0}\,8\,9\,9\,4\,7. \end{array}$$

(2) This operation may be generalised with some modifications for the easy transformation of other fractions also, as the following examples will show :—

(i) $\frac{1}{39} = \begin{array}{l} 1\ \ \ 2210\dot{0} \\ 1.\dot{0}2564\dot{1} \end{array}$ as shown above, 4 being the continued divisor.

(ii) Take $\frac{1}{399}$. Here 40 becomes the continued divisor which is too big for mental division. As there is one zero in the continued divisor, instead of placing each remainder over the preceding digit as in the above case, let us omit one digit and note the remainder over the alternate preceding digit and go on with the process as above, thus :—

$$\frac{1}{399} = \begin{array}{l} \ \ \ \ \ 01202122221020100 0 \\ 01\cdot\dot{0}02506265664160 4\dot{0}\dot{1} \end{array}$$

(iii) Take $\frac{1}{3999}$. Here 400 being the continued divisor, to adjust the operation to this divisor, let us place the remainder over the digit before omitting as many digits as there are zeros in the divisor. Thus :—

$$\frac{1}{3999} = \begin{array}{l} 001{\cdot}2002120221330201230220023210332120 \\ 001{\cdot}\dot{0}002500625156289072268037016754188 \\ 1233103201300012102112230212100012131300 \\ 547136784196049012253063265816454113528 \\ 32301330320011220210020010000 \\ 38209552388097024256064016004000\dot{1}. \end{array}$$

Hence, we see that the process of reducing a fraction into its decimal equivalent becomes easy when the denominator ends with any number of nines preceded by a small number, irrespective of the numerator.

(3) Suppose we have to find the decimal equivalent of $\frac{1}{57}$; let us make the denominator end in nines preceded by a number as small as possible.

$\frac{1}{57} = \frac{7}{399}$, here we can start with 7 and proceed as above.

Further examples :—

$$\frac{1}{31} = \frac{29}{899} \text{ or } \frac{129}{3999};\ \frac{1}{43} = \frac{93}{3999};\ \frac{1}{93} = \frac{43}{3999};\ \frac{1}{47} = \frac{17}{799};$$

$$\frac{1}{17} = \frac{47}{799};\ \frac{1}{131} = \frac{229}{29999};\ \frac{1}{229} = \frac{131}{29999}.$$

Here we see that 43 and 93 mutually co-operate with one another, 43 helping in finding the decimal equivalent of $\frac{1}{93}$ and 93 helping in finding the decimal equivalent $\frac{1}{43}$; similar is the relation between 47 and 17, 131 and 229.

Suppose we have to find out the decimal equivalent of $\frac{24}{31}$; we note

$$\frac{1}{31} = \frac{29}{899};\ \frac{24}{31} = \frac{29 \times 24}{899} \text{ or } \frac{696}{899}.$$

Whenever we have to reduce any fraction to its decimal we must try first to see whether we can reduce it to a denominator containing nines preceded by as small a number as possible. For instance, the decimal equivalent of $\frac{1}{203}$ may be easily found from $\frac{591133}{119999999}$, which is equivalent to $\frac{1}{203}$.

(4) Next, let us take up another interesting process: Take, for instance, any proper fraction, say, $\frac{x}{17}$; here x may be any integral number between 1 and 16 both inclusive.

Place before you the following table :—

1	10	15	14	4	6	9	5
16	7	2	3	13	11	8	12

This table, which will hereafter be called the "*Calculating Table*," presents some peculiar and interesting features. It contains all the numbers from 1 to 16. Each number is ten times the preceding number, 17's being rejected. If we know the first eight numbers in this table, we can find out the remaining eight numbers very easily from the following fact: each pair of numbers of the same order in the two sets amount to the denominator of the given fraction, *i.e*, 17. The first number added to ten times the last number of the first set is the least multiple of 17 which ends in 1, *i.e.*, $51 = 3 \times 17$. Consequently, the first number added to ten times the last number of the second set must as a matter of course be $17 \times (11 - 3)$, which is equivalent to 136. Every number added to ten times the eighth number from it is some multiple of 17.

Now, take any proper fraction with 17 for the denominator, say, $\frac{4}{17}$.

First find with what digit we must multiply the denominator so that the result may end in a nine. The digit is 7. Look for the numerator 4 in the table. Write down all the numbers in the order given in the table beginning with the next number, *viz*, 6.

Thus :—

6, 9, 5, 16, 7, 2, 3, 13, 11, 8, 12, 1, 10, 15, 14, 4.

Then multiply the digit in the units place in each number by the above 7 and note the digit in the units place of each result in order after the decimal point. The result, *viz.* $\cdot\dot{2}35294117647058\dot{8} = \frac{4}{17}$, all the digits recurring.

Section III.

In the previous section, I have mentioned some characteristic properties of the numbers in the Calculating Table (C.T.) and their relation to the decimal expressions themselves. I shall now point out some further curiosities. 17 is a prime number; all the fractions with 17 for the denominator contain sixteen decimal places. The divisors of 16 are 1, 2, 4 and 8. For the present trials, take some fraction, say, $\frac{11}{17}$. Place before you the C. T. for $\frac{11}{17}$ which is 8, 12, 1, 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3, 13, 11 as well as its decimal equivalent which is $\cdot\dot{6}47058823529411\dot{7}$.

Take every first, second, fourth and eighth numbers of the C. T. and total them separately; and side by side take the combinations of 1, 2, 4 and 8 digits of the decimal expression and total them also separately as shown below:—

Divisor 1		Divisor 2		Divisor 4		Divisor 8	
8	6	12	64	10	6470	6	64705882
12	4	10	70	6	5882	11	35294117
1	7	14	58	7	3529		
10	0	6	82	11	4117	17	99999999
15	5	5	35				
14	8	7	29	34	19998		
4	8	3	41				
6	2	11	17				
9	3						
5	5	68	396				
16	2						
7	9						
2	4						
3	1						
13	1						
11	7						
136	72						

By a little examination we see that under divisor 1 the first result is eight times 17 and the second result is eight times one nine; under divisor 2 the first result is four times 17 and the second result is four times the combination of two nines or 99; under divisor 4 the first result is two times 17 and the second result is two times the combination of four nines or 9999; under divisor 8, the first result is 17 and the second result is the combination of eight nines.

SHORT NOTES.

Triangles of Rational Sides and Area.

1. Let ABC be any triangle. Let x, y, z denote $\tan \frac{1}{2} A$, $\tan \frac{1}{2} B$, $\tan \frac{1}{2} C$. We have

$$x = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \ \&c. \quad \ldots \quad \ldots \quad (1)$$

whence

$$\left.\begin{aligned} yz &= \frac{s-a}{s} \\ zx &= \frac{s-b}{s} \\ xy &= \frac{s-c}{s}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Consequently, we have

$$yz + zx + xy = 1, \quad \ldots \quad \ldots \quad (3)$$

and

$$\frac{x(y+z)}{a} = \frac{y(z+x)}{b} = \frac{z(x+y)}{c}, \quad \ldots \quad \ldots \quad (4)$$

each being equal to $1/s$.

2. Let us now suppose the triangle to be of rational sides and area. Then

$$x = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}, \text{ a rational number.}$$

Similarly y and z are rational numbers. Thus, for every triangle whose sides and area are rational, we have rational numbers x, y, z connected by the relation $yz + zx + xy = 1$, such that the sides of the triangle are proportional to

$$x(y+z), \ y(z+x), \ z(x+y).$$

3. Conversely, the theorem is suggested that *if x, y, z be positive rational numbers such that $\Sigma yz = 1$, then the triangle whose sides are*

$$\left.\begin{aligned} a &= kx(y+z) \\ b &= ky(z+x) \\ c &= kz(x+y) \end{aligned}\right\} \ldots \quad \ldots \quad \ldots \quad (5)$$

where k is any rational number, has a rational area.

This is readily verified. For, for this triangle, we have

$$s = k\,(yz + zx + xy) = k,$$
$$s - a = kyz,$$
$$s - b = kzx,$$
$$s - c = kxy,$$
giving,
$$\Delta = k^2\,xyz. \qquad \dots \qquad \dots \quad (6)$$

We see that the theorem embraces all triangles which have rational sides and area.

4. Let us take x, y, z as fixed, and k alone as variable, in formula (5), so that we have a system of *similar* triangles. We proceed to determine the minimum triangle of the system whose sides and area are *integers*.

For this purpose the following simple proposition is useful.

Prop. :—*If the sides and area of a triangle be integers, then, its* s, $(s-a)$, $(s-b)$ *and* $(s-c)$ *are all integers.*

It will be sufficient to show that s is an integer, for the others differ from it by integers. We say s must be an integer. For, if not, $2s$ (which is the integer $a + b + c$) must be an odd number; and therefore also $(2s-2a)$, $(2s-2b)$, $(2s-2c)$ would be odd numbers. Therefore, by multiplication, $16\Delta^2$ must be an odd number; which is absurd since Δ is an integer. Thus, it follows that s, $s-a$, $s-b$, $s-c$ are all integers.

5. Suppose now the triangle given by (5) to have integral sides and area. Its s, $s-a$, $s-b$, $s-c$ are, as already seen, equal to k, kyz, kzx, kxy.

Let yz, zx, xy expressed as fractions in terms of their *least* common denominator be $\frac{l}{\lambda}$, $\frac{m}{\lambda}$, $\frac{n}{\lambda}$. Since $\Sigma yz = 1$, we have $l + m + n = \lambda$. From these expressions for yz, zx, xy we have s, $s-a$, $s-b$, $s-c$ equal to

$$k,\ k\,\frac{l}{\lambda},\ k\,\frac{m}{\lambda},\ k\,\frac{n}{\lambda}.$$

By what is proved in last para. these must be integers. It is at once evident that the least value of k which will render these integrals is $k = \lambda$. This value gives $s = \lambda$, $s - a = l$, $s - b = m$, $s - c = n$; leading to

$$a = \lambda - l = m + n$$
$$b = \lambda - m = n + l$$
$$c = \lambda - n = l + m.$$

We conclude that this is the minimum triangle of the system whose sides and area are integral.

That the area $\triangle$ in this case is integral perhaps requires some explanation. We see it must be rational, for, it is rational for every rational value of k, and therefore for $k = \lambda$. Further, its square is an integer, being $s.(s-a)(s-b)(s-c)$—a product of integers. Therefore $\triangle$ itself must be an integer.

6. Thus, finally, we have the theorem that *if x, y, z are positive rational numbers such that $\sum yz = 1$, and the fractions yz, zx, xy expressed in terms of their least common denominator be l/λ, m/λ, n/λ; then the triangle whose sides are $\lambda - l$, $\lambda - m$, $\lambda - n$, or what is the same, $m+n$, $n+l$, $l+m$, has an integral area, and it is the least triangle of its species whose sides and area are integral.*

V. RAMASWAMI AIYAR.

Note on a Theorem of Mr. Grace.

The object of this note is to give a goemetrical proof of the theorem of Mr. Grace., viz. If β_1, β_2 ... β_r be r *given* distinct linear forms and α_1, α_2 ... α_r *given* positive integers subject to the condition $\sum\alpha = n - r + 1$, then any form of the nth order can be represented uniquely in the shape

$$\sum_{k=1}^{k=r} A_k \, \beta_k^{\,n-\alpha_k}$$

A_p denoting an arbitrary form of order α_p.

1. We have the familiar theorem in space of three dimensions, that a unique straight line can be drawn through any point to meet each of two given non-intersecting straight lines. This theorem may be thus extended to n dimensions:

Let there be r mutually non-intersecting flat regions P_k of α_k dimensions $(k = 1, 2 \ldots r)$ in a space S_n of n dimensions. Further, let the regions P_k be just sufficient to determine the complete n-space, so that $\sum(\alpha_k + 1) = n + 1$, or $\sum\alpha_k = n - r + 1$. Then through any point X we can draw one and only one region of $(r-1)$ dimensions to intersect each of the P_k's.

The proof is quite simple. Since the regions P are just sufficient to determine S_n, we can choose the points of reference for homogenous coordinates x_0 x_1 ... x_n, so that they lie on one or other of the regions P.

Thus the first $(\alpha_1 + 1)$ of the points of reference may be taken to lie on P_1, the next $(\alpha_2 + 1)$ on P_2 and so on. Under this choice of the co-ordinate system, if $x_0\, x_1 \ldots x_n$ be the co-rodinates of X, it is immediately evident that the point X can be expressed as a linear combination of points β_k in P_k $(k = 1, 2 \ldots r)$ in one and only one way; the point β_k being in fact that point in P_k whose non-vanishing co-ordinates are equal to the corresponding co-ordinates of X.

The theorem is thus proved.

2. This theorem would remain valid if the P's are osculating regions of a rational n-ic curve in S_n; for the osculating regions of the rational n-ic are quite general in their intersection relations. Let us now use the representation of binary forms of the nth order by points in the n-space S_n as explained in 'Linear Systems of Points' (*J. I. M. S.*, June 1919). By means of this representation, the point X will correspond to an arbitrary form of the nth order and the points on the rational n-ic will correspond to forms which are perfect nth powers. A point in P_k will correspond to a binary form of the type

$$\beta_k^{\,n-\alpha_k} \, A_k$$

where β_k is the linear form corresponding to the point of osculation and A_k is some form of order α_k. The geometrical theorem proved becomes by this representation the theorem of Mr. Grace.

[For an algebraical proof of the theorem, see *Appendix*, Grace and Young: *Algebra of Invariants.*]

R. VAITHYANATHASWAMI.

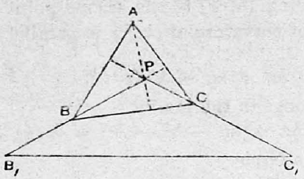
SOLUTIONS.

Question 1075.

(M. K. Kewalramani) :—ABC is a triangle and P is its ortho-centre. PA, PB, PC are produced to A_1, B_1, C_1 respectively, such that $AA_1 = r_1$, $BB_1 = r_2$, $CC_1 = r_3$, where r_1, r_2, r_3 have their usual meanings. If ξ, η, ζ are the normal co-ordinates of the incentre of the triangle ABC with respect to triangle $A_1B_1C_1$, show that

$$\left(s\cos\frac{A}{2}-\xi\right)\sin\frac{A}{2}=\left(s\cos\frac{B}{2}-\eta\right)\sin\frac{B}{2}=\left(s\cos\frac{C}{2}-\zeta\right)\sin\frac{C}{2}=r.$$

Solution by Hemraj.



We have $$PC_1 = PC + CC_1 = \frac{b}{\sin B}\cos C + r_3$$

$$= 2R\left(\cos C + 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= R(1+\cos A+\cos B+\cos C).$$

Similarly $$PB_1 = PA_1 = R(1+\Sigma\cos A).$$

$\therefore$ $$B_1C_1 = 2PB_1\cos\frac{A}{2}.$$

Hence from $\triangle PB_1C_1$, we get $\angle PB_1C_1 = \frac{A}{2}$.

Now the projection of IB_1 on a line perp. to B_1C_1 is equal to the sum of the projections of IB, BB_1 on the same line.

$$\therefore\ \xi = BB_1\sin PB_1C_1 + IB\sin\left(\frac{B}{2}+C+\frac{A}{2}-\frac{\pi}{2}\right)$$

$$= r_2\sin\frac{A}{2} + 2\operatorname{cosec}\frac{B}{2}\sin\frac{C}{2}$$

$$= 4R\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

$$= \sin\frac{A}{2}(4R-r_1) = \sin\frac{A}{2}(r_2+r_3-r)$$

$$= s\cos\frac{A}{2} - r\operatorname{cosec}\frac{A}{2}.$$

Hence the result.

Question 1098 (ii).

(K. J. SANJANA, M.A.) :—If the circle of curvature at a point of the hyperbola $4xy = a^2 + b^2$ touches one of the asymptotes, the point lies at the intersection of the hyperbola with one of the curves

$$3x^4 - 2(a^2 - b^2)\, x^2 + 6x^2y^2 - y^4 = 0.$$
$$3y^4 - 2(a^2 - b^2)\, y^2 + 6y^2x^2 - x^4 = 0.$$

Additional Solution by E. H. Neville.

The tangent at (h, k) to the hyperbola $xy = c^2$ is

$$kx + hy = 2c^2.$$

Hence the line joining (h, k) to the second point of intersection of the curve with its circle of curvature at (h, k) is parallel to $hx + ky = 0$ and is therefore

$$hx + ky = h^2 + k^2.$$

Hence the circle belongs to the family

$$(kx + hy - 2c^2)\,(hx + ky - h^2 - k^2) + \lambda\,(xy - c^2) = 0.$$

For this conic to be a circle,

$$h^2 + k^2 + \lambda = 2hk \cos \omega = c^2 \cos \omega = \tfrac{1}{2}(a^2 - b^2),$$

and for it to touch $y = 0$,

$$\{ k(h^2 + k^2) + 2c^2h \}^2 = 4hkc^2 \{ 2(h^2 + k^2) - \lambda \}.$$

Substituting hk for c^2, and cancelling powers of k, we have λ-eliminant in the form

$$(3h^2 + k^2)^2 = 4h^2 \{ 3(h^2 + k^2) - \tfrac{1}{2}(a^2 - b^2) \},$$

which gives the required result at once.

Question 1116.

(PROF. K. J. SANJANA) :—If r be a positive integer (including zero) such that $(x - r\pi)$ always lies between $-\dfrac{\pi}{2}$ and $+\dfrac{\pi}{2}$, show that

$$\text{(i)} \quad \int_0^\infty \frac{x - r\pi}{x}\, dx = \frac{\pi}{2} \log 2.$$

$$\text{and (ii)} \quad \int_0^\infty (-)^r \frac{x - r\pi}{x}\, dx = 2 \left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} \;\ldots\ldots\ldots\; \textit{ad inf.} \right)$$

Solution by V. M. Gaitonde.

(i) Expand in Fourier's Series

$f(x) \equiv x$ from 0 to $\dfrac{\pi}{2}$, $\equiv x - \pi$ from $\dfrac{\pi}{2}$ to π ; and we get

$$\int_0^\infty \frac{x - r\pi}{x}\, dx = \int_0^\infty \left\{ \sin 2x - \frac{\sin 4x}{2} + \ldots\ldots \right\} \frac{dx}{x}$$
$$= \left\{ 1 - \frac{1}{2} + \frac{1}{3} - \ldots \right\} \frac{\pi}{2}, \text{ for } \int_0^\infty \frac{\sin mx\, dx}{x} = \frac{\pi}{2} = \frac{\pi}{2} \log 2,$$

(ii) Expand in Fourier's Series $f(x) = x$ from 0 to $\frac{\pi}{2}$, and $f(x)$

$= \pi - x$ from $\frac{\pi}{2}$ to π; and we get

$$\int_0^\infty (-1)^r \frac{x - r\pi}{x}\, dx = \frac{4}{\pi} \int_0^\infty \left\{ \frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots \right\} \frac{dx}{x}$$

$$= \frac{4}{\pi} \left\{ \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots \right\} \frac{\pi}{2}$$

$$= 2 \left\{ \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots \right\}.$$

Additional note by " K. B. M."

As an alternative to the above solution, the following *a priori* method may be interesting :—

It is obvious that if $\quad -\frac{\pi}{2} \leq x - r\pi \leq \frac{\pi}{2}$

$$\int_0^\infty \frac{x - r\pi}{x}\, dx$$

$$= \pi \int_0^\infty \frac{x - r}{x}\, dx \text{ where } -\frac{1}{2} \leq x - r \leq \frac{1}{2}$$

$$= \pi \left[\int_0^{\frac{1}{2}} \frac{x}{x}\, dx + \sum_{r=1}^{\infty} \int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{x - r}{x}\, dx \right];$$

and the expression in square brackets above

$$= \lim_{n \to \infty} \left[\frac{1}{2} + \sum_1^n \left(1 - r \log \frac{2r+1}{2r-1} \right) \right]$$

$$= \lim_{n \to \infty} \log \frac{e^{n + \frac{1}{2}}}{(2n+1)^n} \times (3 \cdot 5 \cdot 7 \dots 2n-1).$$

$$= \lim_{n \to \infty} \log \frac{e^{n + \frac{1}{2}} (2n)!}{(2n+1)^n \cdot 2^n \cdot (n)!}$$

$$= \lim_{n \to \infty} \log \sqrt{2} \cdot \frac{e^{\frac{1}{2}}}{\left(1 + \frac{1}{2n}\right)^n}, \text{ using Stirling's theorem}$$

$$= \log \sqrt{2};$$

giving finally the required result $\frac{\pi}{2} \log 2$. Similarly with the second result.

Question 1119.

(N. Dorai Rajan) :—A triangle having its angular points on a limacon has its orthocentre at the node. Show that its circumcircle touches a fixed circle.

Solution by C. N. Sreenivasa Iyengar.

The limacon being the first positive pedal of a circle, if O be the node of the limacon, ABC a triangle inscribed in it so that its orthocentre coincides with O, then the lines thro' A, B, C parallel to BC, CA, AB respectively, are tangents to the above circle. Let these tangents form the Δ DEF. Then the above circle is either the in-circle or one of the ex-circles of Δ DEF, while the circle ABC is the N. P. circle of Δ DEF. Hence, by the well-known theorem of Feuerbach, the circle ABC touches the fixed circle whose first positive pedal is the given limacon.

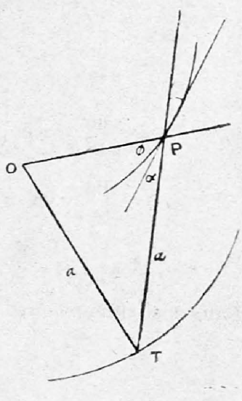
Corollary.—The inverse of a conic with respect to a focus being a limacon, and the inverse of a circle being always a circle, we get the following interesting property of conics: (remembering that if two curves touch, their inverses also touch) :

If a triangle be inscribed in a conic, so as to have its orthocentre at a focus, then the circumcircle touches a fixed circle. A direct proof of this proposition will be interesting.

Question 1121.

(N. Dorairajan) :—Constant lengths a are measured along the tangents to a curve. If the locus of the extremities be a circle of radius a, show that the curve is the inverse of an involute of a circle of radius $2a$, the radius of inversion being $2a$.

Additional solution by A. Narasinga Rao and others.



I shall discuss the more general case where the constant lengths PT are measured off along a line making an angle α with the tangent at P. Let the locus of T be the circle centre O and radius OT $= a$. We have then OT = PT, each being equal to a, so that $\angle POT = \angle OPT = \phi + \alpha$ and we have

$$r = 2a \cos(\phi + \alpha) \qquad \ldots (1)$$

which is the r, ϕ equation of the required curve. Now if we take the inverse of this curve *w. r. t.* a circle of radius b, we have to replace r by $\frac{b^2}{r}$ and leave ϕ unaltered as it is an invariant for inversion. Hence the (r, ϕ) equation of the inverse of (1) is

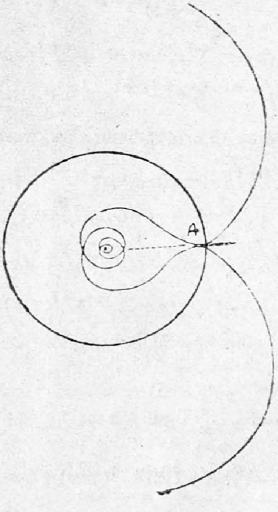
$$b^2 = 2ar \cos(\phi + \alpha). \quad \dots \quad \dots \quad \dots \quad \dots \ (2)$$

But $r \cos(\phi + \alpha)$ is the distance of 0 from one of the α-normals (there are two α-normals at each point one on each side of the normal, except when $\alpha = 0$ or $\frac{\pi}{2}$ when they coincide) of the curve. Hence the curve is the inverse of an α-involute of a circle of radius $(b^2/2a)$ *w. r. t.* another concentric circle of radius b. Putting $\alpha = 0$ and $b = 2a$, we obtain Mr. Dorairajan's result.

If the locus of T be a circle of radius c different from a, we have in the figure OT $= c$ and

$$c^2 = r^2 + a^2 - 2ar \cos \overline{\phi + \alpha}.$$

From the tracing of the curve (for $\alpha = 0$) given below, it will be seen to be a double spiral with O for an asymptotic point and a cusp at A. OA is an axis of symmetry.



Question 1125.

(COMMUNICATED BY G. S. M):—Prove that

$$n!\,D^{-(n+1)}(u)=x^n D^{-1}(u)-c_1x^{n-1}D^{-1}(xu)+\ldots\ldots+(-1)^n D^{-1}(x^n u),$$

where $D\equiv\frac{d}{dx}$ and $u\equiv f(x)$.

Solution by N. Sundaram Aiyar, Martyn M. Thomas, M.A., A. V. Subba Rao and others.

Assume the relation

$$n!\,D^{-(n+1)}(u)=x^nD^{-1}(u)-c_1x^{n-1}D^{-1}(xu)+\ldots\ldots+(-1)^nD^{-1}(x^nu),\ldots\ldots\ldots(1)$$

to be true for a particular value of n.

This can be written

$$(n+1)!\,D^{-(n+1)}(u)=(n+1)x^nD^{-1}(u)$$
$$-{}^{n+1}c_1\,nx^{n-1}D^{-1}(xu)+{}^{n+1}c_2\,.\,(n-1)x^{n-2}D^{-1}(x^2u)+$$
$$\ldots\ldots+(-1)^r\,{}^{n+1}c_r\,.\,(n-r+1)x^{n-r}D^{-1}(x^ru)+$$
$$\ldots\ldots+(-1)^n\,.\,(n+1)D^{-1}(x^nu).$$

Also $0=x^{n+1}\,.\,u-{}^{n+1}c_1\,.\,x^nxu+{}^{n+1}c_2x^{n-1}\,.\,x^2u-\ldots$

$$+(-1)^r\,{}^{n+1}c_rx^{n-r+1}\,.\,x^ru+\ldots\ldots+(-1)^{n+1}(x^{n+1}u).$$

Adding, and grouping the right-hand side, we have

$$(n+1)!\,D^{-(n+1)}(u)=(n+1)x^nD^{-1}(u)+x^{n+1}\,.\,u$$
$$-{}^{n+1}c_1\,\{nx^{n-1}\,.\,D^{-1}(xu)+x^n\,.\,xu\}+\ldots$$
$$+(-1)^r\,{}^{n+1}c_r\{(n-r+1)x^{n-r}D^{-1}(x^ru)+x^{n-r+1}\,.\,x^ru\}$$
$$+\ldots\ldots+(-1)^n\,.\,(n+1)\{D^{-1}(x^nu)+x\,.\,x^nu\}$$
$$+(-1)^{n+1}D^{-1}(x^{n+1}u).$$

Integrating both sides, we have

$$(n+1)!\,D^{-(n+2)}(u)=x^{n+1}D^{-1}(u)-{}^{n+1}c_1\,.\,x^nD^{-1}(xu)+\ldots$$
$$+(-1)^n(n+1)xD^{-1}(x^nu)+(-1)^{n+1}D^{-1}(x^{n+1}u).$$

So, if the relation (1) is true for a particular value of n, it is true for the next higher value.

But $D^{-2}(u) = xD^{-1}(u) - D^{-1}(xu)$, integrating by parts.

Hence the theorem is true for $n=2$, hence for $n=3$, 4, and so on, for all values of n.

Remarks by K. J. Sanjana.

This is a well-known formula and can be readily proved by induction. It is given by Todhunter in the following form :—

$$n!\, u_{n+1} = x^n \int u dx - n x^{n-1} \int x u dx + \frac{n(n-1)}{2!} x^{n-2} \int x^2 u dx \ldots\ldots$$

$$+ (-1)^r \frac{n(n-1)\ldots\ldots(n-r+1)}{r!} x^{n-r} \int x^r u dx + \ldots\ldots + (-1)^n \int x^n u dx.$$

Question 1129.

(S. MALHARI RAO) :—Find a number which when multiplied by 2, 3, 4, 5, 6, 7, 8, or 9 gives in each case a product which contains the same figures as the number itself.

Solution by N. B. Mitra.

We know that if p is a prime of which 10 is a primitive root, then the period of $1/p$ (which will consist of $p-1$ digits) will when multiplied by 2, 3, 4,......$p-1$ consist of the same digits in different cyclic orders.

Similarly the period of a/p, where a is $< p$, will, when multiplied by 2, 3, 4,...x, where ax is $< p$, consist of the same digits as the period itself.

In the present case p must be greater than 9 and the smallest prime greater than 9 of which 10 is a primitive root is 17. Hence the period of 1/17 when multiplied by 2, 3, 4,...16 will consist of the same digits in different cyclic orders. But the period of 1/17 begins with a zero (as will the periods of all fractions $1/p$, where p is >9). Hence if we seek a number which does not begin with zero we must choose a in a/p in such a way that a/p is not less than 1/10 and $9a$ is not greater than $p-1$, a being less than p which must be a prime and must have 10 as a primitive root.

The fraction with the smallest denominator, satisfying these conditions is 2/19 whose period is 105263157894736842 and this number when multiplied by 2, 3, 4,...9 will consist of the same figures in different cyclic orders. The next number is the period of 3/29, *i.e.* 1034482758620689655172413793. And the next numbers are the periods of 5/47, 6/59, 10/97, etc. which consist of 46, 58, 96 digits respectively.

Question 1130.

(A. A. KRISHNASWAMI AIYANGAR) :—Prove that a triangle ABC in which $\tan A = \tan^{2n+1} B$ and $\tan 2B = m \tan C$ where m, n are positive integers is right-angled and isosceles.

(1) *Solution by N. B. Mitra.*

It is easy to see that the relations

$\tan A = \tan^{2n+1} B$, ... (1)

$\tan 2B = m \tan C$, ... (2)

$A + B + C = 180°$, ... (3)

are satisfied by $A = B = 45°$ and $C = 90°$.

We shall show that there are no other solutions.

If $B < 45°$ then $\tan 2B$ is positive and therefore by (2) $\tan C$ is positive, so that $C < 90°$.

$\therefore$ $A > 45°$ and $\tan A$ is > 1 ; but $\tan B < 1$; hence (1) will not hold.

Again if B is $> 45°$ and < 900 C must be $> 90°$ by (2) since $\tan 2B$ is negative in this case. Hence A is $< 45°$ and $\tan A$ is < 1, whereas $\tan B > 1$ and so (1) will not hold good.

If $B > 90°$, $A < 90°$.

$\therefore$ $\tan B$ is negative and $\tan A$ is positive $\therefore$ (1) will not hold.

Hence the only solution is $B = 45°$, therefore from (1) $A = 45°$ and also $C = 90°$.

(2) *Solution by L. N. Subramaniam and several others.*

Obviously $\tan C = -\tan(A+B)$.

Hence we have

$$\tan 2B = \frac{2 \tan B}{1 - \tan^2 B} = -m \tan(A+B)$$

$$= \frac{-m(\tan^{2n+1} B + \tan B)}{(1 - \tan^{2n+2} B)},$$

$\therefore$ $2(1 - \tan^{2n+2} B) + m(1 + \tan^{2n} B)(1 - \tan^2 B) = 0$,

i.e. $(1 - \tan^2 B)\{2(1 + \tan^2 B + \tan^4 B + \ldots\ldots \tan^{2n} B)$
$+ m(1 + \tan^{2n} B)\} = 0$.

All the terms in the second factor here are all positive, as m is a positive integer, hence $(1 - \tan^2 B) = 0$,

i.e. $\tan B = \pm 1$, and from the given data, the + sign alone is admissible, as $\tan A \neq -1$, as also $\tan A = -1$, if $\tan B = -1$.

Hence, we have $A = B = 45°$, *i.e.*, the $\triangle$ is rt. angled and isosceles.

Question 1131.

(A. A. KRISHNASWAMI AIYANGAR):—Solve in positive integers the equation $$9x^4 - 516x^2y^2 + 7168y^4 = z^2.$$

Solution by N. B. Mitra.

We have $9x^4 - 516x^2y^2 + 7168y^4 = z^2$ ... ... (1)

Suppose $z = 3x^2 - 2ky^2$ ... ... (2)

From (1) $x^2(3k - 129) = y^2(k^2 - 1792)$, so that, we may take

$$x^2 = k^2 - 1792 \text{ and } y^2 = 3k - 129 = 3(k - 43) \quad (3)$$

Let $k = 43 + n$, then $x^2 = n^2 + 86n + 57 = (n + m)^2$, suppose; whence $n = \dfrac{57 - m^2}{2(m - 43)}$ and m must be odd, if n is to be an integer.

Let $m = 2p + 1$, then $n = -p - 22 - \dfrac{448}{p - 21}$. ... (4)

Let $448 = aa'$, where a and a' are integers + or —.

Then $p = a + 21$ is a solution of (4).

Hence $n = -(a + a' + 43)$, $m = 2a + 43$, $k = -(a + a')$.

From (3) $n = \pm(a - a')$ and $y^2 = -3(a + a' + 43)$.

Hence $(a + a' + 43)$ must be negative and therefore a and a' must be negative. Now the only possible values of a and a' are the following pairs:—

$-1, -448$; $-2, -224$; $-4, -112$; $-7, -64$; $-8, -56$; $-14, -32$; $-16, -28$; and it will be found that the only admissible pair, which will make $-3(a + a' + 43)$ a perfect square, is $-14, -32$. Taking these as the values of a and a', we get $x = 18$, $y = 3$, $z = 144$; or rejecting a common factor, we get the only solutions of the type given by (2), viz., $x = 6$, $y = 1$, $z = 16$.

Or we may proceed thus:—

Equation (1) may be written $(3x^2 - 86y^2)^2 - 228y^4 = z^2$.

Assume $3x^2 - 86y^2 = u$, $2y^2 = v$ so that $x^2 = (u + 43v)/3$, $y^2 = v/2$ and our equation becomes $u^2 - 57v^2 = z^2$ or $(u + z)(u - z) = 57v^2$.

Now $57v^2$ can be broken up into factors either as $19v \cdot 3v$, or $57v \cdot v$.

Taking the first case, we may take $u + z = 19v$, $u - z = 3v$, which lead to $x^2 = 18v$, $y^2 = v/2$, v, $z = 8v$. Taking $v = 2a^2$, we get $x = 6a$, $y = a$, $z = 16a^2$, which is the same solution as obtained by the first method.

If we take the second pair of factors, it will be found that $x^2 = 24v$, so that $x^2y^2 = 12v^2$, which does not give rational values for x and y.

Question 1132.

(N. Dorairajan):—If O, I, H be the circum-in and ortho-centres of a triangle ABC, and if the circle OIH passes through one angular point, prove that it also passes through another angular point. Find the necessary condition. (Suggested by Q. 1037).

Solution by S. Purushotham.

We know AI bisects OAH. Similarly BI bisects OBH and CI bisects OCH.

Suppose AHIO are concyclic, then HI = IO.

Then in the Δs BHI and BIO, BI is common, $\angle HBI = \angle OBI$, and $HI = OI$;

$\therefore$ angles BHI and BOI are equal or supplementary.

Similarly, angles CHI and COI are equal or supplementary.

If $\angle BHI$ equals $\angle BOI$ then $BH = BO$;

that is $2R \cos B = R. \qquad \therefore B = 60^\circ.$

If $\angle BHI$ and $\angle BOI$ are supplementary, circle HOI passes through B.

B and C cannot both be 60°, for then the triangle would be equilateral.

So the circle HOI passes through A and either B or C, since it cannot pass through both B and C. Suppose it passes through B.

Then $\angle CHI$ and $\angle COI$ cannot be supplementary. $\therefore$ they are equal

$\therefore CH = CO$, *i.e*, $2R \cos C = R. \qquad \therefore C = 60^\circ.$

This is easily seen to be a sufficient condition also.

Note.—In the given question O and H may be any two isogonal conjugates. The proposition would still be true though, of course, the condition would be different. In this general case, the triangle may be any triangle but O or H should lie on one of the circles BIC, AIB and CIA.

Question 1133.

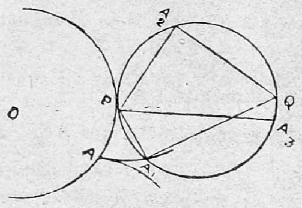
(N. Dorairajan):—Show that in an epi-cycloid where the radius of the fixed circle is n (integer) times that of the rolling circle, the feet of the normals to the curve from any point on the fixed circle form the angular points of a regular polygon. Show that a nearly similar theorem holds for the tangents.

Solution and remarks by A. Narasinga Rao, Martyn M. Thomas, and others.

Consider the more general case of two circles whose circumferences are ml and nl respectively (m and n being integers prime to each other),

the former rolling on the latter. Then if we mark off from a fixed pt. A, of the rolling circle, lengths nl, $2nl$, $3nl$,......$(m-1)nl$, we obtain m.-pts. A_2......A_m, which are the vertices of a regular m-gon inscribed in the circle. Now let rolling commence with A_1 in contact with the fixed circle at A so that there is a cusp at A for the hypocycloid. Since the lengths $A_1 A_2$, $A_2 A_3$, etc. are each equal to the circumference of the fixed circle, the pts. A_2, A_3......will pass in succession through the pt. A *and will therefore describe the same curve*, namely the locus of A_1. Similarly, if we mark off lengths AB, BC,......along the circumference of the fixed circle each $= nl$ we obtain n distinct pts. which the pts. with which A_1 comes into contact in succession, *i.e.*, the cusps on the epicycloid. Hence we have :—

If the ratio of the fixed to the rolling circle be as $m:n$ (m and n being relatively prime integers) the epi-cycloid described has n cusps arranged symmetrically on the circumference, and covers the circumference m times over. Also there exists a regular m-gon every vertex of which describes the same epi-cycloid.



The instantaneous centre in any position is the pt. of contact P; hence the normals at A_1 A_2...A_m are respectively A_1P, A_2P, ... A_mP so that the feet of the normals from any pt. P of the fixed circle form the corners of a regular polygon. Similarly, the tangents at $A_1 A_2$ ...... A_m meet at the pt. Q diametrically opposite to P, so that the tangents from any pt. on a concentric circle of radius=twice the sum of the two radii, have their pts. of contact at the vertices of a regular polygon.

These considerations fail when $m=1$ but it is well known that an epi-cycloid can be generated in 2 ways and for the other method of generation $m=n+1$. This is the case covered by Mr. Dorairajan's question.

These tangents (or normals) meet at Q (or P) at angles which are multiples of $\frac{\pi}{m}$, so that a circle is part of the isoptic locus when the angle of intersection is $\frac{\pi}{m}$ or any multiple thereof. In particular, when m is even, one of these multiples is $\frac{\pi}{2}$, so that the circle counts $\frac{m}{2}$ times as part of the orthoptic locus.

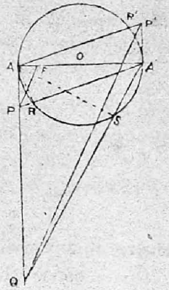
Question 1136.

(Selected by T. Krishna Rao):—Parallel lines through the foci of an ellipse meet the tangent at the vertex A in P, Q and the lines joining P, Q to the other vertex A′ meet the circle on AA′ as diameter again in R and S. Prove geometrically that RS is a tangent to the ellipse.

Solution by S. Purushotham and J. B. Freeman.

Let T be the point on the auxiliary circle diametrically opposite S.

Let F and F′ be the foci.



Let RT cut axis in Y.

Then $$\frac{AT \,.\, AR}{A'T \,.\, A'R} = \frac{\perp \text{ from A on RT}}{\perp \text{ from A}' \text{ on RT}} = \frac{AY}{YA'}.$$

But $$\frac{AR}{A'R} = \frac{AP}{AA'}$$ by similar triangles ARA′ and APA′

and $$\frac{AT}{A'T} = \frac{A'S}{AS} = \frac{AA'}{AQ}.$$

$$\therefore \quad \frac{AR}{A'R} \cdot \frac{AT}{A'T} = \frac{AP}{AQ}.$$

Again $$\frac{AP}{AQ} = \frac{AF}{AF'},$$ since APF and AQF′ are similar triangles.

$$\therefore \quad \frac{AY}{YA'} = \frac{AF}{AF'} = \frac{AF}{FA'}.$$

Hence Y and F coincide.

$$\therefore \quad FR \,.\, FT = b^2.$$

$$\therefore \quad FR \,.\, F'S = b^2.$$

Hence RS is a tangent to the ellipse.

Similar solutions by N. Sundaram Aiyar and several others.

Question 1139.

(R. VAITHYANATHASWAMY) :—If $x_0\ x_1 \ldots\ldots x_n$ are the homogeneous co-ordinates in n dimensions, find the complete curve of intersection of the following surfaces :—

$$x_0\,x_2 - x_1{}^2 = 0$$
$$x_1\,x_3 - x_2{}^2 = 0$$
$$\ldots \quad \ldots \quad \ldots$$
$$x_{n-2}\ x_n - x^2{}_{n-1} = 0.$$

Solution by A. Narasinga Rao.

The common points of all the surfaces are given by

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \ldots\ldots = \frac{x_{n-1}}{x^n} = \frac{1}{t}\,(\text{say}), \quad \ldots \quad \ldots \quad (1)$$

so that, we may take

$$x_0 = 1 \quad x_1 = t \quad x_2 = t^2 \ldots\ldots x_n = t^n, \quad \ldots \quad \ldots \quad (2)$$

as the parametric equations of the curve.

The curve is readily seen to be a twisted n-ic, the analogue in n-space to the plane conic and the twisted cubic of 3-space. It is curve of degree n and class n-ic ; any plane cuts the curve in n points and through a given point pass n osculating planes (spaces represented by linear equations). If any n-ad of elements $\lambda_1\ \lambda_2 \ldots \lambda_n$ be the roots of an equation

$$\sum_0^n x_r\,\lambda^r = 0,$$

and be represented in n-space by the point whose homogeneous co-ordinates are $x_0\ x_1 \ldots x_n$, the equations (1) are satisfied by coincident n-ads ($\lambda_1 = \lambda_2 \ldots = \lambda_n$) and the twisted n-ic may be studied as the locus of coincident n-ads. For a discussion of its properties by this method, the reader is referred to Mr. Vaithyanathaswamy's article on "Linear Systems of Points," Vol. XI.

Question 1147.

(V. RAMASWAMI AIYAR) :—If A, B, C, D be four concyclic points, prove that the in-and ex-centres of the triangles ABC, BCD, CDA, DAB lie on a quartic which passes through A, B, C, D and touches the circle ABCD at the circular points at infinity.

If the equations of the circle and the rectangular hyperbola through ABCD be $x^2 + y^2 = R^2$ and $xy + \lambda x + \mu y + \gamma = 0$, show that the

equation of the quartic is

$$\frac{x^2 + y^2 - R^2}{2R^2} = \frac{xy + \lambda x + \mu y + \gamma}{xy - \gamma}$$

Solution by S. Purushotham.

We shall prove the second part first. Any conic passing through A, B, C, D is given by $kx^2 + ky^2 - kR^2 + 2xy + 2\lambda x$

$$+ 2\mu y + 2\gamma = 0. \quad \ldots \quad (1)$$

Let $(x - g)^2 + (y - f)^2 - r^2 = 0 \quad \ldots \quad (2)$

be the equation of any one of the in- or ex-circles of one of the 4 given triangles. Then any triangle circumscribed about (2) can be inscribed in (1)

Therefore by invariants

$$\theta'^2 = 4\,\Delta'\,\theta.$$

$$\Delta' = -r^2,\ \theta' = 2\,(fg + \lambda g + \mu f + \gamma) + k\,(f^2 + g^2 - 2r^2 - R^2)$$

$$\theta = (-\mu^2 - \lambda^2 - f^2 - g^2 + r^2 - 2f\lambda - 2g\mu) + k\,(4\gamma + 2f\mu + 2g\lambda)$$
$$+ k^2\,(-2R^2 + f^2 + g^2 - r^2).$$

In the relation $\theta'^2 = 4\Delta'\theta$, equate the co-efficients of k on both sides.

Then

$$4\,(fg + \lambda g + \mu f + \gamma)\,(f^2 + g^2 - 2r^2 - R^2)$$
$$= -\,4r^2\,(4\gamma + 2f\mu + 2g\lambda).$$

$$\therefore\ \frac{(fg + \lambda g + \mu f + \gamma)}{\lambda g + \mu f + 2\gamma} = \frac{-\,2r^2}{f^2 + g^2 - 2r^2 - R^2},$$

$$\therefore\ \frac{fg + \lambda g + \mu f + \gamma}{fg - \gamma} = \frac{+\,2r^2}{f^2 + g^2 - R^2}.$$

But since $SI^2 = R^2 \pm 2Rr$,

$$(f^2 + g^2 - R^2) = \pm\ 2Rr$$

$$\therefore\ \frac{2r^2}{f^2 + g^2 - R^2} = \frac{f^2 + g^2 - R^2}{2R^2}.$$

$$\therefore\ \frac{fg + \lambda g + \mu f + \gamma}{fg - \gamma} = \frac{f^2 + g^2 - R^2}{2R^2},$$

which is the required result.

The first part of the question easily follows from this.

The quartic passes through A, B, C, D.

If $x = \pm\,(y)$ in the equation of the quartic, we get only 2nd degree terms, that is the circular lines cut the quartic in two points at infinity, showing that the quartic touches circle at infinity.

QUESTIONS FOR SOLUTION.

1185. (V. RAMASWAMI AIYAR):—Find how rational values may be assigned, in a most general manner, to the edges of a tetrahedron so that the areas of the faces, and the volume, may all be rational.

1186. (Communicated by P. V. SESHU AIYAR):—If

$$z = \operatorname{Exp}\left[-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x{}^2}-\frac{2rxy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y{}^2}\right)\right]$$

and

$$p_{s,} = \frac{1}{\sigma^s{}_x\,\sigma^t{}_y}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^s\, y^t\, z\, dxdy,$$

show that

(1) $p_{s,t} = 0$, if $s + t$ is odd ;

(2) $p_{s,t} = (s + t - 1)\, rp_{s-1,\,t-1}$
$$+ (s-1)(t-1)(1-r^2)\, p^2{}_{s-2,\,t-2} ;$$

(3) $p_{s,\,t} = (t-1)\, p_{s,\,t-2} + srp_{s-1,\,t-1}$
$$= (s-1)\, p_{s-2,\,t} + trp_{s-1,\,t-1} ;$$

(4) $$p_{2s,\,2t} = \frac{2s\,!\;2t\,!}{2^{s+t}}\sum_{u=0}^{u=t}\frac{(2r)^{2u}}{(s-u)\,!\,(t-u)\,!\,2u\,!} ;$$

(5) $$p_{2s+1,\,2t+1} = \frac{(2s+1)\,!\;(2t+1)\,!r}{2^{s+t}}\sum_{u=0}^{u=t}\frac{(2r)^{2u}}{(s-a)\,!\,(t-u)\,!\,(2u+1)\,!}.$$

1187. (F. H. V. GULASEKHARAM):—If f, g, h be the lengths of the bisectors of the angles A, B, C respectively of a triangle ABC, prove

(i) $$[g^2 + h^2 - 2gh\cos\tfrac{1}{2}(B-C)]^{\frac{1}{2}} + [h^2 + f^2 - 2hf\cos\tfrac{1}{2}(C-A)]^{\frac{1}{2}} + [f^2 + g^2 - 2fg\cos\tfrac{1}{2}(A-B)]^{\frac{1}{2}} = 0;$$

(ii) $$\sin\tfrac{1}{2}(B-C)/f + \sin\tfrac{1}{2}(C-A)/g + \sin\tfrac{1}{2}(A-B)/h = 0.$$

1188. (F. H. V. GULASEKHARAM):—Solve the equations

$$\sin(2x + y) = a\sin 2x,$$
$$\sin(x + 2y) = b\sin 2y.$$

1189. (J. B. FREEMAN):—From a point P tangents are drawn to a given ellipse and to all confocal coins. Show that the locus of the points of contact of these tangents is the same as the locus of the foot of the perpendicular drawn from P upon the chord of contact. Explain the significance of this property.

1190. (N. P. PANDYA) :—It is required to illustrate by putting together a minimum number of card-board pieces, that the complements of a parallelogram are equal. The point through which the parallels are drawn is at a distance of 1·5 in. from the vertex on the longer diagonal of a parallelogram whose angle is 60° and whose sides are 4 in. and ·6 in. respectively. Give a description of all pieces required for this purpose.

1191. (G. V. TELANG) :—ABC is a triangle inscribed in a circle. Three tangents are drawn to the circle so that the portions of them cut off by AB and AC are bisected at the points of contact P, Q and R. Show that the orthocentre of the triangle PQR lies at the mid. point of BC.

1192. (R. VAIDYANATHASWAMI):—If $(S+\lambda S')$ is a pair of lines for the values a, a^{-1}, $-a^{-1}$ of λ, prove that quadrilaterals can be inscribed in $S-aS'$ with their four sides touching respectively the conics $(S+bS')$, $(S-bS')$, $(S+b^{-1}S')$, $(S-b^{-1}S')$

1193. (R. VAIDYANATHASWAMI) :—E is a conic of a given four-point system and S_1, S_2 are the two conics of the system which are inscribed to a self-conjugate Δ of E. Triangles are inscribed in E with two of their sides touching S_1, S_2 respectively. Prove that the envelope of the third side is composed of the two conics of the system each of which has triangles inscribed in itself, circumscribed to the other.

1194. (PROF. SANJANA) :—If $f(x)$ denote

$$x^n+2p_1x^{n-1}+2^2p_2x^{n-2}+\ldots\ldots+2^{n-1}p_{n-1}x+2^n p_n,$$

prove that the result of eliminating x from the two equations

$$f(y)+\frac{x^2}{2!}f_2(y)+\frac{x^4}{4!}f_4(y)+\ldots\ldots=0,$$

$$f_1(y)+\frac{x^2}{3!}f_3(y)+\frac{x^4}{5!}f_5(y)+\ldots\ldots=0,$$

is the same as that of eliminating $y_2, z_1, z_2 \ldots\ldots z_{n-2}$ from the following n equations—

$$y+z_1+p_1=0,\ yz_1+y_2+z_2-p_2=0,$$
$$yz_2+y_2z_1+z_3+p_3=0,\ \ldots\ldots\ldots\ldots,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$yz_{n-2}+y_2z_{n-3}-(-1)^{n-1}p_{n-1}=0,\ y_2z_{n-2}-(-1)^n p_n=0.$$

[$f_1, f_2, f_3 \ldots\ldots$ denote derived functions of f with respect to y.]

THE INDIAN MATHEMATICAL SOCIETY.

Summary of Accounts for the year 1920.

Receipts :—	Rs.	A.	P.
Balance from 1919 ...	6529	15	1
Subscriptions from Members ...	785	0	0
Subscriptions for Journal ...	88	4	0
Interest on Investments ...	188	9	2
	7591	12	3

Expenditure :—	Rs.	A.	P.
Ordinary working expenses ...	171	5	3
Books and Journals ...	1051	1	6
Printing Journal ...	497	0	0
Library expenses ...	468	8	0
Closing balance ...	5403	13	6
	7591	12	3

The Closing Balance is distributed as follows:—

	Rs.	A.	P.
Madras Central Urban Bank,	2000	0	0
Indian Bank, Madras ...	1949	7	4
Bombay Savings Bank ...	400	0	0
Treasurer ...	797	7	11
Secretary ...	71	13	6
Assistant Secretary ...	53	14	6
Assistant Librarian ...	113	2	0
Joint Editor ...	18	0	3
TOTAL ...	5403	13	6

MADRAS,
17th September 1921.

S. NARAYANA AIYAR,
Hon. Treasurer.